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Michael Aschbacher

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WALTER'S BASIC THEOREM FOR FUSION SYSTEMS

Michael Aschbacher

California Institute of Technology

Abstract. This is the first of two papers determining the saturated 2-fusion systems in which the centralizer of some fully centralized involution contains a component that is the 2-fusion system of a large group of Lie type over a field of odd order.

The results in this paper are part of a program to, first, classify (essentially) the simple 2-fusion systems of component type, and then, second, to use the theorem on fusion systems to simplify the proof of the theorem classifying the finite simple groups. See [A4] and [A5] for a discussion of the program.

Let p be a prime and S a finite p-group. A *fusion system* on S is a category $\mathcal{F}$ whose objects are the subgroups of S and, for subgroups P, Q of S, the set $\hom_{\mathcal{F}}(P, Q)$ of morphisms from P to Q is a set of injective group homomorphisms from P to Q, and that set satisfies two weak axioms. The standard example is the fusion system $\mathcal{F}_S(G)$ for G a finite group and $S \in Syl_p(G)$, whose morphisms are those induced via conjugation in G. A fusion system is *saturated* if it satisfies two more axioms easily seen to hold in the standard example using Sylow's Theorem. See [AKO] for notation, terminology, and basic definitions and results on fusion systems.

Let $\mathcal{F}$ be a saturated fusion system on a finite 2-group S. Call S the *Sylow group* of $\mathcal{F}$ and write $S \in Syl(\mathcal{F})$. Proceeding by analogy with finite groups, one can define the notion of a *normal subsystem* of $\mathcal{F}$, which can then be used to define the notions of *simple* and *quasisimple* systems, *subnormal subsystems* of $\mathcal{F}$, and the set $\mathrm{Comp}(\mathcal{F})$ of *components* of $\mathcal{F}$. For t an involution in S the *centralizer* $C_{\mathcal{F}}(t)$ of t in $\mathcal{F}$ is defined, and if t is *fully centralized* (ie. $|C_S(t)| \geq |C_S(x)|$ for each conjugate x of t) then $C_{\mathcal{F}}(t)$ is saturated, so we can define $\mathrm{Comp}(C_{\mathcal{F}}(t))$.

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Define $\mathfrak{C}(\mathcal{F})$ to be the set of *components of centralizers of involutions* in $\mathcal{F}$; that is $\mathcal{C} \in \mathfrak{C}(\mathcal{F})$ if there exists some involution $t \in S$ and a conjugate $(\bar{t}, \bar{\mathcal{C}})$ of $(t, \mathcal{C})$ such that $\bar{t}$ is fully centralized and $\bar{\mathcal{C}} \in \mathrm{Comp}(C_{\mathcal{F}}(\bar{t}))$; we write $\mathcal{I}(\mathcal{C})$ for the set of such involutions t. We say that $\mathcal{F}$ is of *component type* if $\mathfrak{C}(\mathcal{F})$ is nonempty.

Let $\mathcal{K}$ be the class of "known" simple 2-fusion systems, and $\tilde{\mathcal{K}}$ the class of "known" quasisimple 2-fusion systems: those whose central factor system is in $\mathcal{K}$. In our attempt to classify the simple systems of component type, we proceed inductively, and hence assume each member of $\mathfrak{C}(\mathcal{F})$ is in $\tilde{\mathcal{K}}$.

The largest and most interesting class of known quasisimple systems of component type are those of the form $\mathcal{F}_S(G)$ for $G \in Chev^*(p)$ for some odd prime p and $S \in Syl_2(G)$. Here $Chev^*(p)$ consists of the quasisimple groups of Lie type and characteristic p, other than the Ree groups and the groups $L_2(q)$. We say such a group G is defined over $\mathbf{F}_q$ if its fundamental subgroups are isomorphic to $SL_2(q)$, and we write $Chev^*[m]$ for the class of 2-fusion systems of groups in $Chev^*(p)$ defined over $\mathbf{F}_q$ where $m = (q^2-1)_2$ is the 2-share of $q^2 - 1$.

Generically, the class of the simple systems in $Chev^*[m]$ is the class of systems $\mathcal{F}$ in $\mathcal{K}$ such that $\mathfrak{C}(\mathcal{F})$ contains a member of $Chev^*[m]$. There are a few members of $Chev^*[8]$ that are not of component type, while if $\mathcal{F}$ is an exotic Benson-Solomon system $\mathcal{F}_{Sol}(q)$ then $\mathfrak{C}(\mathcal{F})$ contains the 2-fusion system of $Spin_7(q)$. This paper is a step in the direction of a proof of what we call Walter's Theorem for Fusion Systems, which determines the simple 2-fusion systems $\mathcal{F}$ for which $\mathfrak{C}(\mathcal{F})$ contains a large member of $Chev^*[m]$ for some m.

Write $SL_2[m]$, $Spin_7[m]$ for the 2-fusion system of $SL_2(q)$, $Spin_7(q)$ with $(q^2-1)_2 = m$, respectively. Write $\hat{A}_n$ for the universal covering group of A_n.

A *solvable component* of $\mathcal{F}$ is a subnormal subsystem of $\mathcal{F}$ isomorphic to the 2-fusion system of $SL_2(3)$ or $L_2(3)$. Write $\mathrm{Comp}_+(\mathcal{F})$ for the union of $\mathrm{Comp}(\mathcal{F})$ with the set of solvable components of $\mathcal{F}$. Write $\mathfrak{C}_+(\mathcal{F})$ for the set of conjugates of members $\mathcal{C}$ of $\mathrm{Comp}_+(C_{\mathcal{F}}(t))$ for t a fully centralized involution in $\mathcal{F}$.

A member of $\mathcal{L}$ of $\mathfrak{C}_+(\mathcal{F})$ is *intrinsic* in $\mathfrak{C}_+(\mathcal{F})$ if $\mathcal{I}(\mathcal{L}) \cap Z(\mathcal{L}) \neq \varnothing$.

Finally $Chev[large]$ is a certain subset of $\mathcal{K}$ (defined in Definition 3.1) consisting of $Chev^*[m]$ for all $m > 8$ and almost all of $Chev^*[8]$.

The main theorem of this paper is a fusion theoretic version of a result of John Walter in [W] on groups, and is a key step in the proof of Walter's Theorem.

Theorem. *(Walter's Basic Theorem) Assume $\mathcal{F}$ is a saturated fusion system on a finite*

2-group S such that each member of $\mathfrak{C}(\mathcal{F})$ is in $\tilde{\mathcal{K}}$. Assume $\mathfrak{C}(\mathcal{F})$ contains a member of $Chev[large]$. Then there is $\mathcal{L} \in \mathfrak{C}_+(\mathcal{F})$ intrinsic in $\mathfrak{C}_+(\mathcal{F})$ such that one of the following holds:

(1) $\mathcal{L}$ is $SL_2[m]$ for some $m \geq 8$.

(2) $\mathcal{L}$ is $Spin_7[m]$ for some $m \geq 8$.

(3) $\mathcal{L}/Z(\mathcal{L})$ is the 2-fusion system of $L_3(4)$ and $\mathcal{I}(\mathcal{L}) \cap \Phi(Z(\mathcal{L})) \neq \varnothing$.

(4) $\mathcal{L}$ is the 2-fusion system of $\hat{A}_n$ for some even $n \geq 8$.

If $\mathcal{F}$ is simple and $\mathfrak{C}_+(\mathcal{F})$ contains an intrinsic member isomorphic to $SL_2[m]$ then $\mathcal{F}$ is in $Chev^*[m]$ by the Main Theorem of [A6]. Thus to prove Walter's Theorem, it suffices to analyze cases (2), (3), and (4) of Walter's Basic Theorem. For example in case (2) we want to prove that $\mathcal{F}$ is a Benson-Solomon system when $\mathcal{F}$ is simple.

For j an involution in S, we often write $\mathcal{F}_j$ for $C_{\mathcal{F}}(j)$.

Section 1. Preliminary results

Basic notation, terminology, definitions, and results on fusion systems can be found in [AKO]. Our basic reference on finite groups is [FGT]. Our notation for listing the 2-fusion systems in $Chev^*[m]$ can be found in the Table in section 1.1 of [A6]; for example $G_2[m]$ is the 2-fusion system of $G_2(q)$ for $m = (q^2-1)_2$. There is more discussion of the notation for the systems of orthogonal groups in 1.1.

The notion of a *tame realization* of a fusion system by a group is defined and discussed in section 3.3 of [AO]. By Theorems 3.5 and 3.6 in [AO], the members of $\mathcal{K}$, other than the exotic Benson-Solomon systems, are tamely realized by some known simple group, and hence by Theorem 2.20 in [AO], each of its coverings is tamely realized by a known quasisimple group.

Notation 1.1. Let V be an n-dimensional orthogonal space over a finite field $F = \mathbf{F}_{q_0}$, where q_0 is an odd prime power and $n \geq 5$. Let ϵ be the sign of V if n is even. Set $\Omega = \Omega(V)$. Then there is a pairing (K, K') of fundamental subgroups of Ω given by $[K, K'] = 1$ and $[V, K] = [V, K']$.

Let $\tilde{\Omega}$ be the universal cover of Ω and $\tilde{Z} = Z(\tilde{\Omega})$. Thus $\tilde{\Omega} = Spin_n^{\epsilon}(q_0)$, and if $\tilde{K}$ is the inverse image of K in $\tilde{\Omega}$ then $\tilde{K} = J\tilde{Z}$, where $J = O^2(\tilde{K}) \cong K \cong SL_2(q_0)$. In particular J contains a unique involution $z(J)$ and setting $J' = O^2(\tilde{K}')$, $b = z(J)z(J')$ is an involution in $\tilde{Z}$ independent of the choice of fundamental subgroup K, since Ω is transitive on its fundamental subgroups and respects the pairing.

If $n \equiv 0 \mod 4$ and $\epsilon = +1$ then $\tilde{Z} \cong E_4$ and we write $HSpin^+_n(q_0)$ for either of the factor groups $\tilde{\Omega}/A$ with A a subgroup of $\tilde{Z}$ of order 2 distinct from $\langle b\rangle$. Write $Spin^\delta_n[m]$, $HSpin^+_n[m]$ for the 2-fusion system of $Spin^\epsilon_n(q_0)$, $HSpin^+_n[q_0]$, respectively, where $m = (q_0^2 - 1)_2$ and $\delta = \pm 1$ is a function of $\pm 1 = \pi \equiv q_0 \mod 4$, ϵ, and n described in the Table in section 1.1 of [A6].

(1.2) *Assume $\mathcal{F}$ is a saturated fusion system on a finite 2-group S and $\mathcal{D}$ is a normal quasisimple subsystem of $\mathcal{F}$. Assume t is a fully centralized involution in S and $\mathcal{C} \neq \mathcal{D}$ is a component of $C_{\mathcal{D}}(t)$ such that $\mathcal{C} \cong SL_2[m]$ for some $m \geq 16$ and $Z(\mathcal{C}) \leq Z(\mathcal{D})$. Assume $\mathcal{D}$ is tamely realized by some known quasisimple group L. Then one of the following holds:*

(1) $\mathcal{D}$ is isomorphic to $Spin^\epsilon_n[m]$ or $Spin^\epsilon_n[m/2]$ for some $n \geq 5$.

(2) $\mathcal{D}$ is isomorphic to $HSpin^+_n[m]$ or $HSpin^+_n[m/2]$ for some $n \equiv 0 \mod 4$ with $n \geq 8$.

(3) $\mathcal{D} \cong SL_2[2m]$ and t induces a field automorphism on $\mathcal{D}$.

(4) $m = 16$, $\mathcal{D}/Z(\mathcal{D})$ is the 2-fusion system of $L_3(4)$, $Z(\mathcal{C}) \leq \Phi(Z(\mathcal{D}))$, and t induces a field automorphism on $\mathcal{D}/Z(\mathcal{D})$.

(5) $m = 16$ and $\mathcal{D}$ is the 2-fusion system of $\hat{A}_n$ for some even $n \geq 8$.

Proof. By hypothesis, $\mathcal{D}$ is tamely realized by some quasisimple group L. Set $L^* = L/Z(L)$ and let T be Sylow in $\mathcal{C}$. Then by 2.22 in [AO], t acts on L and as $\mathcal{C} \neq \mathcal{D}$, t is faithful on L^*. As $\mathcal{C} \in \mathrm{Comp}(C_{\mathcal{D}}(t))$, it follows from 2.5.11 in [A5] that $\mathcal{C} = \mathcal{F}_T(K)$ for some component K of $C_L(t)$. As $\mathcal{C} \cong SL_2[m]$, $K \cong SL_2(q)$ for some prime power q such that $(q^2 - 1)_2 = m$, or $m = 16$ and $K \cong \hat{A}_7$. Set $a = z(\mathcal{C})$. Then $\langle a\rangle = Z(K)$ and by hypothesis, $a \in Z(L)$, so $K^* \cong L_2(q)$ or A_7.

Suppose first that $L^* \cong A_n$. Then as $K^* \in \mathrm{Comp}(C_{L^*}(t^*))$, $K^* \cong A_6$ or A_7, so in particular $m = 16$. As $K < L$, we conclude that $n \geq 8$. As $L^* \cong A_n$ and $1 \neq a \in Z(L)$, it follows that $L \cong \hat{A}_n$. Therefore (5) holds as $\mathcal{D}$ is also the 2-fusion system of $\hat{A}_{n-1}$ when n is odd.

Suppose next that $L^* \in Chev(2)$. We may assume L^* is not $L_4(2)$ as $L_4(2) \cong A_8$. As $1 \neq a \in Z(L)$ it follow from Table 6.1.3 in [GLS3] that L^* is one of the groups listed in that table, or equivalently in 6.4.3 of [A5]. As $K^* \in \mathrm{Comp}(C_{L^*}(t^*))$, t^* induces an outer automorphism on L^*, so t^* and $C_{L^*}(t^*)$ are described in section 19 of [ASe]. In particular $K^* \in Chev(2)$, so $K^* \cong L_3(2)$ or $A_6 \cong Sp_4(2)'$. We conclude from [ASe] that $(L^*, K^*) = (L_3(4), L_3(2))$ and t induces a field automorphism on L^*. As $a \in K$ it follows that for each involution i^* in L^*, there is a preimage i of i^* of order 4 with $i^2 = \bar{a}$. As there is also an involution preimage, $\bar{a} \in \Phi(Z(L))$. Hence (4) holds in this case.

Suppose that L^* is sporadic. Again it follows from Table 6.1.3 in [GLS3] that L^* is one of the groups listed there, or equivalently in 6.4.2 in [A5]. As $K^* \in \mathrm{Comp}(C_{L^*}(t^*))$ we conclude from [GLS3] that (L^*, K^*) is $(J_2, L_2(7))$ or (HS, A_6). But then from [GLS3], $Z(K) = 1$, a contradiction.

Therefore we may assume that $L^* \in Chev(p)$ for some odd prime p. Suppose first that L^* is exceptional. Then as $Z(L)$ is of even order, we conclude from section 6.1 of [GLS3] that $L^* \cong E_7(q_0)$. But now it follows from Table 4.5.1 in [GLS3] that L^* admits no involutory automorphism t whose centralizer has a component isomorphic to $L_2(q)$.

Therefore L^* is classical. Here there is an n-dimensional vector space V over $F = \mathbf{F}_{q_0}$ such that L^* is an image of $\Omega = L^\epsilon(V)$, $Sp(V)$, or $\Omega^\epsilon(V)$. Let $\tilde{\Omega}$ be the universal 2-covering group of Ω.

Suppose first that V is orthogonal, so that we can adopt Notation 1.1, and in particular $\tilde{\Omega} = Spin_n^\epsilon(q_0)$. From section 6.1 in [GLS3], $|\tilde{\Omega}| = 2|\Omega|$, and setting $\gamma = |\tilde{Z}|$, one of the following holds:

(i) $\gamma = 2$ and either n is odd or $n \equiv 0 \mod 4$ and $\epsilon = -1$.

(ii) $\tilde{Z} \cong \mathbf{Z}_d$, $n \equiv 2 \mod 4$, and $d = (4, q_0 - \epsilon)$.

(iii) $\tilde{Z} \cong E_4$, $n \equiv 0 \mod 4$, and $\epsilon = +1$.

The conjugacy classes of involutions in $Aut(L^*)$ are listed in Table 4.5.1 in [GLS3] and in section 15 of [A7]. In particular as $C_{L^*}(t^*)$ has a component isomorphic to $L_2(q)$, we conclude that either t^* is of type $i(3)$ or $i(4,-1)$, or $(n, \epsilon) = (8, 1)$ and t^* is conjugate under a graph automorphism of order 3 to such an involution. For example if n is odd this follows from 15.3.2 and 15.11 in [A7], so take n even. Here 15.11 in [A7] does not quite suffice as there are involutions in $Aut(L^*)$ inducing similarities but not isometries on V. However from Table 4.5.1 in [GLS3] there are no such involutions whose centralizer has an $L_2(q)$-component, unless $(n, \epsilon) = (8, 1)$, where all such involutions are fused in $Aut(L^*)$ to isometries. Thus in any event t^* is fused into $i(3)$ or $i(4,-1)$, so K^* is $L_2(q_0)$ or $L_2(q_0^2)$, respectively. In the first case, $q = q_0$, while in the second $q = q_0^2$.

Suppose $\gamma = 2$. Then as $Z(L) \neq 1$ it follows that $L = Spin_n^\epsilon(q_0)$, so as $q = q_0$ or q_0^2, $\mathcal{D}$ is $Spin_n^\epsilon[m]$ or $Spin_n^\epsilon[m/2]$, respectively. Therefore (1) holds in this case, so we may assume $\gamma > 2$, and hence (ii) or (iii) holds, with $\tilde{Z} \cong \mathbf{Z}_4$ or E_4 in the respective case.

Next in either case, $L = \tilde{\Omega}/A$ for some $A \leq \tilde{Z}$ with $a = bA$ for some $b \in \tilde{Z} - A$. If $A = 1$ then $L = \tilde{\Omega} = Spin_n^\epsilon(q_0)$, and then, arguing as in the previous paragraph, (1) holds. Therefore we may assume $A \neq 1$, so $|A| = 2$.

Let $\tilde{K}$ be the preimage in $\tilde{\Omega}$ of K and set $K_0 = E(\tilde{K})$. Then $K_0 \cong K \cong SL_2(q)$ and we may take b to generate $Z(K_0)$. As $b \notin A$, $\tilde{Z} = \langle b \rangle \times A \cong E_4$, so case (iii) holds.

Suppose that t^* is in $i(3)$ or $i(4,-1)$. From 1.1, $\Omega = \tilde{\Omega}/B$ for a distinguished subgroup B of $\tilde{Z}$ of order 2. Now $[V, \tilde{K}]$ is of dimension $r = 3$ or 4, and the image of $\tilde{K}$ in Ω is isomorphic to $\mathbf{Z}_2 \times L_2(q)$, so we conclude that $B = \langle b \rangle$. Then as $A \cap B = 1$, L is $HSpin_n^+(q_0)$, so (2) holds in this case.

Therefore we may assume that t^* is not in $i(3)$ or $i(4,-1)$, so from earlier discussion, $n = 8$. Hence a graph automorphism of $\tilde{\Omega}$ is transitive on the three subgroups of $\tilde{Z}$ of order 3, so once again $L \cong HSpin_8^+(q_0) \cong \Omega_8^+(q_0)$. This completes the proof in the case where V is orthogonal.

Therefore L^* is $L_n^\epsilon(q_0)$ or $PSp_n(q_0)$. In each case $\Omega = \tilde{\Omega}$; that is $\tilde{\Omega}$ is $SL_n^\epsilon(q_0)$ or $Sp_n(q_0)$.

If $n = 2$ then $L^* \cong L_2(q_0)$, so $L = \tilde{\Omega} \cong SL_2(q_0)$ and (3) holds. Thus we may assume that $n > 2$. As $Z(L)$ is of even order, $n \neq 3$, so $n \geq 4$. Then as $L_4^\epsilon(q_0) \cong P\Omega_6^\epsilon(q_0)$ and $PSp_4(q_0) \cong \Omega_5(q_0)$, we may assume that $n \geq 5$. But now by inspection of Table 4.5.1 in [GLS3], there is no involutory inner-diagonal or graph automorphism i of L^* such that $C_{L^*}(i)$ has a component isomorphic to $L_2(q)$. This contradiction completes the proof of the lemma.

(1.3) *Assume $\mathcal{F}$ is a saturated fusion system on a finite 2-group S and $F^*(\mathcal{F}) = Z(\mathcal{F})\mathcal{D}$ for some quasisimple subsystem $\mathcal{D}$ of $\mathcal{F}$ on D. Assume for some $m \geq 8$ and $i = 1, 2$ that $\mathcal{C}_i \cong SL_2[m]$ is a subsystem of $\mathcal{F}$ on $T_i \leq D$, and, setting $z_i = Z(\mathcal{C}_i)$, that $z_1 \in Z(\mathcal{F})$ $z = z_1z_2 \notin Z(\mathcal{F})$ is fully centralized, and $\mathcal{C}_i \in \mathrm{Comp}_+(C_{\mathcal{F}}(z))$. Then $\mathcal{D} \cong Spin_7[m]$.*

Proof. As $z_1 \in Z(\mathcal{F})$, for $z \in P \leq C_S(z))$ and $\phi \in \hom_{\mathcal{F}}(P, S)$, ϕ lifts to $\varphi \in \hom_{\mathcal{F}}(\langle P, z_1 \rangle, S)$ fixing z_1, and $z\phi = z\varphi = (z_1z_2)\varphi = z_1 \cdot z_2\varphi$. Thus $z = z\phi$ iff $z_2 = z_2\varphi$, so $\mathcal{F}_z = C_{\mathcal{F}}(z) = C_{\mathcal{F}}(z_2) = \mathcal{F}_2$, Then $\mathcal{C}_i \in \mathrm{Comp}_+(\mathcal{F}_z) = \mathrm{Comp}_+(\mathcal{F}_2)$.

Set $\Omega = T_2^{\mathcal{F}}$. As $SL_2[m] \cong \mathcal{C}_2 \in \mathrm{Comp}_+(\mathcal{F}_2)$ with $z_2 \in \mathcal{C}_2$, it follows from 2.4.1 in [A6] that $\tau = (\mathcal{F}, \Omega)$ is a quaternion fusion packet. Then as $T_2 \leq \mathcal{D}$ and $z_2 \notin Z(\mathcal{D})$ (since $z \notin Z(\mathcal{F})$), it follows from the Main Theorem of [A6] that τ is the Lie packet of $\mathcal{D} = \mathcal{F}_D(L)$ for some $L \in Chev^*(p)$ and some odd prime p.

Suppose first that $m > 8$. Then as in the proof of 1.2, $\mathcal{C}_i = \mathcal{F}_{T_i}(K_i)$ for some $K_i \in \mathrm{Comp}(C_L(z_2))$ with $K_2 \cong SL_2(q)$ and $K_1 \cong SL_2(q_0)$ for some q, q_0 with $(q_0^2-1)_2 = m = (q^2-1)_2$. Indeed by 1.2, either

(a) $\mathcal{D} \cong Spin_n^\epsilon[m]$ or $Spin_n^\epsilon[m/2]$, or

(b) $\mathcal{D} \cong HSpin_n^+[m]$ or $HSpin_n^+[m/2]$ for some $n \equiv 0 \mod 4$.

Then as $\mathcal{C}_1 \in \mathrm{Comp}(\mathcal{F}_2)$ with $z_1 \in Z(\mathcal{D})$ and $(q_0^2-1)_2 = (q^2-1)_2$, it follows that $\mathcal{D} \cong Spin_7[m]$, so the lemma holds in this case.

Therefore we may assume $m = 8$, and hence take $L = X(3)$ to be defined over $\mathbf{F}_3$ and $\mathcal{C}_i$ to be the 2-fusion system of $SL_2(3)$ with $\mathcal{C}_2 = \mathcal{F}_{T_2}(K_2)$ for some fundamental subgroup K_2 of L. As $z_1 \in Z(L)$, we conclude as in the proof of 1.2 that L is classical or the covering group of $E_7(3)$. Then as K_2 is a fundamental subgroup of L and $\mathcal{C}_1 \cong SL_2(3)$ is in $\mathrm{Comp}_+(\mathcal{F}_2)$ with $z_1 \in Z(L)$, it follows again that $\mathcal{D} \cong Spin_7[8]$. This completes the proof of the lemma.

Section 2. Walter's trick

In this section we proof the following theorem:

Theorem 2.1. *Assume $\mathcal{F}$ is a saturated fusion system on a finite 2-group S. Assume for some $m \geq 8$ and $i = 1, 2$ that $\mathcal{C}_i \cong SL_2[m]$ is a subsystem of $\mathcal{F}$ on T_i. Set $z_i = z(\mathcal{C}_i)$ and assume $z = z_1z_2$ is a fully centralized involution with $\mathcal{C}_i \in \mathrm{Comp}_+(C_{\mathcal{F}}(z))$. Assume $z_1 \in Z(\mathcal{F})$ Then either*

(1) $\mathcal{C}_1 \in \mathrm{Comp}_+(\mathcal{F})$, or

(2) there exists a component $\mathcal{D} \cong Spin_7[m]$ of $\mathcal{F}$ containing $\mathcal{C}_i$ for $i = 1, 2$.

The proof of Theorem 2.1 involves a series of reductions. Thus throughout this section we assume the hypothesis of Theorem 2.1. Set $\mathcal{F}_z = C_{\mathcal{F}}(z)$ and $\mathcal{F}_2 = C_{\mathcal{F}}(z_2)$.

(2.2) *(1) $\mathcal{F}_z = \mathcal{F}_2$.*

(2) $z_2 \in \mathcal{F}^f$.

(3) For $i = 1, 2$, $\mathcal{C}_i \in \mathrm{Comp}_+(\mathcal{F}_2)$.

Proof. Parts (1) and (3) follow from the argument in paragraph one of the proof of 1.3. Further this argument shows that $\mathfrak{A}(z_2) = \mathfrak{A}(z)$, so (2) follows as $z \in \mathcal{F}^f$.

(2.3) *Assume $m > 8$. Then*

(1) For $i = 1, 2$, $\mathcal{C}_i \leq \mathcal{L}_i$ for some $\mathcal{L}_i \in \mathrm{Comp}(\mathcal{F})$.

(2) Theorem 2.1 holds in this case.

Proof. As $m > 8$, $\mathcal{C}_i$ is quasisimple. Thus $\mathcal{C}_i \leq E(\mathcal{F})$ by E-balance (cf Theorem 7 in [A3]). In particular $z_2 \in \mathcal{C}_2 \leq E(\mathcal{F})$, so z_2 acts on each component of $\mathcal{F}$, and then (1) follows from 2.2.2 and 10.11.3 in [A3].

Suppose z_2 centralizes $\mathcal{L}_1$. Then by 2.2.3 in [A6] and 2.2.3, $\mathcal{L}_1 = \mathcal{C}_1$, so that conclusion (1) of Theorem 2.1 holds in this case. Thus we may assume z_2 does not centralize $\mathcal{L}_1$, so $\mathcal{L}_1 = \mathcal{L}_2 = \mathcal{D}$. Then the hypothesis of 1.3 are satisfied by $\mathcal{D}$ in the role of $\mathcal{F}$, so, we conclude from 1.3 that conclusion (2) of Theorem 2.1 holds in this case. This completes the proof.

(2.4) *Assume $m = 8$ and $\mathcal{C}_2$ does not centralize $E(\mathcal{F})$. Then*

(1) there exists a unique component $\mathcal{L}$ of $\mathcal{F}$ such that $\mathcal{C}_2$ does not centralize $\mathcal{L}$. Moreover $T_2 \leq L \in Syl(\mathcal{L})$.

(2) If T_1 does not centralize $\mathcal{L}$ then $\mathcal{C}_i \leq \mathcal{L}$ for $i = 1, 2$ and $\mathcal{L} \cong Spin_7[8]$.

(3) If T_1 centralizes $\mathcal{L}$ then $\mathcal{C}_1 \in \mathrm{Comp}_+(\mathcal{F})$.

Proof. We conclude from 2.2.7 in [A6] applied to $z_2, \mathcal{C}_2$ in the role of $z, \mathcal{C}$ that (1) holds, and, defining $\mathcal{Y}$ as in that lemma, $\mathcal{C}_2 \leq \mathcal{Y}$ with $F^*(\mathcal{Y}) = Z(\mathcal{Y})\mathcal{L}$ and $C_2 \in \mathrm{Comp}_+(C_{\mathcal{Y}}(z_2))$.

Let L be Sylow in $\mathcal{L}$. By 2.2.6 in [A6], either T_1 centralizes $\mathcal{L}$ or $T_1 \leq L$. Suppose $T_1 \leq L$. Then T_1 centralizes a Sylow group S_0 of $\mathcal{F}_0 = C_{F^*(\mathcal{F})}(\mathcal{L})$ and $\mathcal{F}_0 \leq \mathcal{F}_2$, so γ of order 3 in $Aut_{\mathcal{C}_1}(T_1)$ centralizes S_0 by 2.2.1.3 in [A6]. Then using 9.5.1 in [A3], $\mathcal{C}_1$ centralizes $\mathcal{F}_0$, so $\mathcal{C}_1 \leq \mathcal{Y}$. But now applying 1.3 to $\mathcal{Y}$ in the role of $\mathcal{F}$, we conclude that (2) holds.

Therefore we may assume that T_1 centralizes $\mathcal{L}$. Let $\alpha \in \mathfrak{A}(T_1)$. By 2.2.5.3 in [A6] applied to $\mathcal{C}_1$ in the role of $\mathcal{C}$, $\mathcal{C}_1\alpha^* \trianglelefteq N_{\mathcal{F}}(T\alpha)$, so $\mathcal{C}_1$ centralizes $\mathcal{L}$. Arguing as in the previous paragraph, $\mathcal{C}_1$ centralizes each component of $\mathcal{F}$ distinct from $\mathcal{L}$, so $\mathcal{C}_1 \leq \mathcal{E} = C_{\mathcal{F}}(E(\mathcal{F}))$. Further $\mathcal{E} \leq \mathcal{F}_2$, so $\mathcal{C}_1$ is a solvable component of $\mathcal{E}$. Then as $\mathcal{E} \trianglelefteq \mathcal{F}$, $\mathcal{C}_1$ is a solvable component of $\mathcal{F}$, establishing (3).

(2.5) *Assume $m = 8$ and $\mathcal{C}_2$ centralizes $E(\mathcal{F})$. Set $\tilde{\mathcal{F}} = C_{\mathcal{F}}(E(\mathcal{F}))$. Then*

(1) $\mathcal{C}_1 \leq \tilde{\mathcal{F}}$.

(2) $\tilde{\mathcal{F}}$, $\mathcal{C}_1$, $\mathcal{C}_2$ satisfies the hypothesis of Theorem 2.1 in the role of $\mathcal{F}$, $\mathcal{C}_1$, $\mathcal{C}_2$.

(3) If $\mathcal{C}_1 \in \mathrm{Comp}_+(\tilde{\mathcal{F}})$ then $\mathcal{C}_1 \in \mathrm{Comp}_+(\mathcal{F})$.

(4) $\tilde{\mathcal{F}}$ is constrained.

Proof. Part (4) is 9.12.3 in [A3].

Let E be Sylow in $E(\mathcal{F})$. Then $E(\mathcal{F}) = E(\mathcal{F}_2)$ by 10.3 in [A3]. Therefore (1) follows from 2.2.3 and 2.2.2 in [A6]. By Theorem 4 in [A3], $\tilde{\mathcal{F}} \trianglelefteq \mathcal{F}$, so by 8.23.2 in [A3], $\mathcal{E} = C_{\tilde{\mathcal{F}}}(z) \trianglelefteq \mathcal{F}_z$. Let $\mathcal{X}_i$ be the product of the conjugates of $\mathcal{C}_i$ in $\mathcal{F}_z$. As $\mathcal{E} \trianglelefteq \mathcal{F}_z$, we have $\mathcal{X}_i \leq \mathcal{E}$, and then as $\mathcal{X}_i \trianglelefteq \mathcal{F}_z$, it follows from 3.6.1 in [A1] that $\mathcal{X}_i$ is weakly normal in $\mathcal{E}$. Let $X_i \in Syl(\mathcal{X}_i)$; by 2.2.1.3 in [A6], $\mathcal{X}_i$ centralizes $C_S(X_i)$, so $\mathcal{X}_i \trianglelefteq \mathcal{E}$. Therefore $\mathcal{C}_i \in \mathrm{Comp}_+(\mathcal{E})$, so (2) follows, using 3.4.5 in [A1] to conclude that $z \in \tilde{F}^f$.

As $\tilde{\mathcal{F}} \trianglelefteq \mathcal{F}$, (3) holds.

(2.6) *If $\mathcal{F}$ is constrained then $\mathcal{C}_1 \in \mathrm{Comp}_+(\mathcal{F})$.*

Proof. Assume $\mathcal{F}$ is constrained. Then $E(\mathcal{F}) = 1$, so $m = 8$ by E-balance. Set $\Omega = T_2^{\mathcal{F}}$. By 2.2.3 and 2.4.1 in [A6], $\tau = (\mathcal{F}, \Omega)$ is a quaternion fusion packet. Hence we adopt the notation of [A6] in discussing τ. In particular by 3.1.5 in [A6], $O = \langle \Omega \rangle \trianglelefteq S$ is a central product of the members of Ω and $\mathcal{E} = N_{\mathcal{F}}(O)$ is transitive on Ω. Then by 2.2.1.3 in [A6], $\mathcal{C}_2 \leq \mathcal{E}$. As $\mathcal{F}$ is constrained and $S \in Syl(\mathcal{E})$, $\mathcal{E}$ is also constrained, so $\mathcal{E}$ has a model G. Set $G_2 = N_G(T_2)$. By 2.2.5.1 in [A6], there is $H \trianglelefteq G_2$ with $SL_2(3) \cong H$ a model for $\mathcal{C}_2$. As O is a central product of the members of Ω, $O \leq G_2$ and then by 2.2.5.3 in [A6], $C_S(T_2) = C_S(H)$, so $O = T_2 C_O(H)$. Therefore $K = \langle H^G \rangle$ is a central product of the members of $\mathcal{H} = H^G$.

Set $Q = O_2(\mathcal{F})$. Then $Q \leq S \leq G$ and $[Q, K] \leq Q \cap K = O_2(K) = O$, so as K is a central product of the members of $\mathcal{H}$, Q acts on each member of Ω. Therefore Q acts on H, so as H is irreducible on $T_2/Z(T_2)$, either $T_2 \leq Q$ or $T_2 \cap Q = Z(T_2)$. But in the latter case $[H, Q] \leq C_H(T_2) = Z(T_2)$, so $H \leq C_G(Q) = Z(Q)$, a contradiction. Therefore $T_2 \leq Q$, so $\Omega = T_2^{\mathcal{F}} \subseteq Q$.

We've shown that Q acts on each member of Ω and hence centralizes z_2. By Theorem 2.8.11 in [A5], the kernel $\mathcal{N}$ of the action of $\mathcal{F}$ on Ω is normal in $\mathcal{F}$. Recall $Q \leq \mathcal{F}_2$, so by 2.2.4.4 in [A6] applied to $\mathcal{C}_1 \in \mathrm{Comp}_+(\mathcal{F}_2)$, $QT_1 = T_1 C_Q(\mathcal{C}_1)$. Therefore as $Aut_{\mathcal{C}_1}(T_1)$ is irreducible on $T_1/Z(T_1)$, either $T_1 \leq Q$ or $T_1 \cap Q = Z(T_1)$, and in the latter case $\mathcal{C}_1 \leq C_{\mathcal{F}}(Q) = Z(Q)$, a contradiction. Therefore $T_1 \leq Q = T_1 C_Q(\mathcal{C}_1)$, so for $\gamma \in Aut_{\mathcal{C}_1}(T_1)$ of order 3, we have $|Q : C_Q(\gamma)| = 4$. Then as γ permutes Ω, γ acts on each member of Ω, so $\mathcal{C}_1 \leq \mathcal{N}$. Then as $\mathcal{N} \leq \mathcal{F}_2$, $\mathcal{C}_2 \in \mathrm{Comp}_+(\mathcal{N})$, so as $\mathcal{N} \trianglelefteq \mathcal{F}$, also $\mathcal{C}_1 \in \mathrm{Comp}_+(\mathcal{F})$, completing the proof.

We are now in a position to complete the proof of Theorem 2.1. Assume $\mathcal{F}$ is a counter example. Then $m = 8$ by 2.3.2. Then by 2.4, $\mathcal{C}_2 \leq \tilde{\mathcal{F}} = C_{\mathcal{F}}(E(\mathcal{F}))$. By 2.5, replacing $\mathcal{F}$ by $\tilde{\mathcal{F}}$ if necessary, we may assume $\mathcal{F}$ is constrained. Then 2.6 supplies a contradiction, completing the proof of Theorem 2.1.

Section 3. Walter's basic theorem

Definition 3.1. Given $m = 2^e \geq 8$, define $Chev^*[m]$ to consist of the 2-fusion systems of quasisimple groups G in $Chev^*(p)$ for some odd prime p for which a fundamental subgroup of G is isomorphic to $SL_2(q)$ with $(q^2-1)_2 = m$. A system $\mathcal{F} \in Chev^*[m]$ is *small* if $m = 8$ and $\mathcal{F}$ is $L_3^\epsilon[8]$, $G_2[8]$, $P\Omega_n^\epsilon[8]$ with $5 \le n \le 8$, or $\Omega_6^+[8]$. Define $\mathcal{F}$ to be *large* if $\mathcal{F}$ is not small. Write $Chev[small]$, $Chev[large]$ for the small, large members of $Chev^*[m]$, as m ranges over all powers of 2 of size at least 8.

Definition 3.2. For $\mathcal{F} \in Chev^*[m]$ with Sylow group S, $\mathcal{F}$ is tamely realized by some $G \in Chev^*(p)$ by 3.5 in [AO]. The *fundamental subsystems* of $\mathcal{F}$ are the subsystems $\mathcal{C}$ of $\mathcal{F}$ such that $\mathcal{C} = \mathcal{F}_T(K)$ for some fundamental subgroup K of G such that $T = K \cap S \in Syl_2(K)$. Write $z(\mathcal{C})$ for the involution in T. Write $\mathfrak{L}(G,S)$ for the set of fundamental subsystems of $\mathcal{F}$ and set

$$\Omega(G,S) = \{T : T \in Syl(\mathcal{C}) \text{ for some } \mathcal{C} \in \mathfrak{L}(G,S)\}.$$

Thus $\tau(G,S) = (\mathcal{F}, \Omega(G,S))$ is the *Lie packet* of $\mathcal{F}$.

From [A6], K is subnormal in $C_G(z(\mathcal{C}))$ and we may choose $z(\mathcal{C}) \in \mathcal{F}^f$; thus $\mathcal{C} \in \mathrm{Comp}_+(C_{\mathcal{F}}(z(\mathcal{C}))$.

We often abuse notation and write $\mathfrak{L}(\mathcal{F})$ for $\mathfrak{L}(G,S)$.

The main result of this paper is:

Theorem 3.3. *Assume $\mathcal{F}$ is a saturated 2-fusion system on a finite 2-group S and each member of $\mathfrak{C}(\mathcal{F})$ is in $\tilde{\mathcal{K}}$. Assume some member of $\mathfrak{C}(\mathcal{F})$ is in $Chev[large]$. Then there exists $\mathcal{L} \in \mathfrak{C}_+(\mathcal{F})$ intrinsic in $\mathfrak{C}_+(\mathcal{F})$ such that one of the following holds:*

(1) $\mathcal{L}$ is $SL_2[m]$ for some $m \ge 8$.

(2) $\mathcal{L}$ is $Spin_7[m]$ for some $m \ge 8$.

(3) $\mathcal{L}/Z(\mathcal{L})$ is the 2-fusion system of $L_3(4)$ and $\mathcal{I}(\mathcal{L}) \cap \Phi(Z(\mathcal{L})) \neq \varnothing$.

(4) $\mathcal{L}$ is the 2-fusion system of $\hat{A}_n$ for some even $n \ge 8$.

Thus in the remainder of this section we assume:

Hypothesis 3.4. Assume $\mathcal{F}$ is a saturated 2-fusion system on a finite 2-group S and each member of $\mathfrak{C}(\mathcal{F})$ is in $\tilde{\mathcal{K}}$.

Notation 3.5. Write $\mathfrak{P}(m)$ for the set of pairs $(\mathcal{C}_1, \mathcal{C}_2)$ such that $SL_2[m] \cong \mathcal{C}_i \le \mathcal{F}$ and, setting $z_i = z(\mathcal{C}_i)$ and $z = z_1 z_2$, z is a fully centralized involution in $\mathcal{F}$ such that for $i = 1, 2$, $z_i \in \mathcal{F}_z^f$ and $\mathcal{C}_i \in \mathrm{Comp}_+(C_{\mathcal{F}_z}(z_i))$, where $\mathcal{F}_z = C_{\mathcal{F}}(z)$.

Recall from section 1.2 in [A6] that for $P \leq S$, $\mathfrak{A}(P)$ consists of those $\alpha \in \hom_{\mathcal{F}}(P,S)$ such that $P\alpha \in \mathcal{F}^f$, and that various properties of this notation are listed in 1.2.5 in [A6].

Notation 3.6. (The *bar setup* and *bar notation*) Assume t and a are commuting involutions in S with $t \in \mathcal{F}^f$ and $a \in \mathcal{F}_t^f$. Let $\alpha \in \mathfrak{A}(a)$ and for $U \subseteq C_S(a)$, set $\bar{U} = U\alpha$. Set $\bar{\mathcal{F}} = C_{\mathcal{F}}(\bar{a})$ and for $\mathcal{C} \leq C_{\mathcal{F}_t}(a)$ set $\bar{\mathcal{C}} = \mathcal{C}\alpha^*$.

(3.7) *Assume the bar setup of 3.6. Then*

(1) $\bar{t} \in \bar{\mathcal{F}}^f$ and $\alpha : C_{\mathcal{F}_t}(a) \to C_{\bar{\mathcal{F}}}(\bar{t})$ is an isomorphism of fusion systems.

(2) If $\mathcal{C} \in \mathrm{Comp}_+(C_{\mathcal{F}_t}(a))$ then $\bar{\mathcal{C}} \in \mathrm{Comp}_+(C_{\bar{\mathcal{F}}}(\bar{t}))$.

Proof. Part (1) follows from 2.2 in [A2]. Then (1) implies (2).

(3.8) *Assume the bar setup of 3.6 with $\mathcal{C} \in \mathrm{Comp}(C_{\mathcal{F}_t}(a))$. Then one of the following holds:*

(1) $\bar{\mathcal{C}} \in \mathrm{Comp}(\bar{\mathcal{F}})$.

(2) There exists a component $\mathcal{D}$ of $\bar{\mathcal{F}}$ such that $\bar{t}$ is nontrivial on $\mathcal{D}$, $\bar{t}$ is fully centralied in $\langle \bar{t} \rangle \mathcal{D}$, and $\bar{\mathcal{C}} \in \mathrm{Comp}(C_{\mathcal{D}}(\bar{t}))$.

(3) There exists a component $\mathcal{D}$ of $\bar{\mathcal{F}}$ such that $\mathcal{D} \neq \mathcal{D}^{\bar{t}}$ and $\bar{\mathcal{C}} = E(C_{\mathcal{D}\mathcal{D}^{\bar{t}}}(\bar{t}))$ is a morphic image of $\mathcal{D}$.

Proof. See the proof of 6.1.11 in [A5], except for the assertion in (2) that $\bar{t}$ is fully centralized in $\mathcal{Y} = \langle \bar{t} \rangle \mathcal{D}$. Let E, D be Sylow in $E(\bar{\mathcal{F}})), \mathcal{D}$, respectively. Then E is strongly closed in $\bar{\mathcal{F}}$, so as $\bar{t} \in \bar{\mathcal{F}}^f$, we have $|C_E(\bar{t})| \geq |C_E(s)|$ for each $s \in \bar{t}^{E(\bar{\mathcal{F}})}$, so $\bar{t}$ is fully centralized in $\langle \bar{t} \rangle E(\bar{\mathcal{F}})$. Similarly as D is strongly closed in $E(\bar{\mathcal{F}})$, $\bar{t} \in \mathcal{Y}^f$.

(3.9) *Assume $(\mathcal{C}_1, \mathcal{C}_2) \in \mathfrak{P}[m]$ and adopt the bar notation with $(t,a) = (z, z_1)$. Then either*

(1) $\bar{\mathcal{C}}_1 \in \mathrm{Comp}_+(\bar{\mathcal{F}})$, or

(2) $\langle \bar{\mathcal{C}}_1, \bar{\mathcal{C}}_2 \rangle \leq \mathcal{D} \in \mathrm{Comp}(\bar{\mathcal{F}})$ with $\mathcal{D} \cong Spin_7[m]$.

Proof. By construction, $\bar{z}_1 \in Z(\bar{\mathcal{F}})$. As $(\mathcal{C}_1, \mathcal{C}_2) \in \mathfrak{P}[m]$, we have $\mathcal{C}_i \in Comp_+(C_{\mathcal{F}_z}(z_i))$ for $i = 1, 2$. But as $z \in Z(\mathcal{F}_z)$ with $z = z_1 z_2$, $C_{\mathcal{F}_z}(z_1) = C_{\mathcal{F}_z}(z_2)$, so for $i = 1, 2$, $\mathcal{C}_i \in \mathrm{Comp}_+(C_{\mathcal{F}_z}(z_1))$. Then by 3.7, the hypotheses of Theorem 2.1 are satisfied by $\bar{\mathcal{F}}, \bar{\mathcal{C}}_1, \bar{\mathcal{C}}_2, \bar{z}, \bar{z}_1, \bar{z}_2$ in the role of $\mathcal{F}, \mathcal{C}_1, \mathcal{C}_2, z, z_1, z_2$. Now Theorem 2.1 completes the proof.

(3.10) *Assume z is an involution in $\mathcal{F}^f$ and $\mathcal{L}$ is a component of $\mathcal{F}_z$ such that $z \in Z(\mathcal{L})$ and one of the following holds:*

(a) $\mathcal{L} \cong Spin_n^\epsilon[m]$ for some $n \geq 5$ and $m \geq 8$. Further if $n \equiv 0 \mod 4$ and $\epsilon = 1$ then z is the distinguished involution b of 1.1.

(b) $\mathcal{L} \cong HSpin_n^+[m]$ for some $n \geq 8$ with $n \equiv 0 \mod 4$.

(c) $\mathcal{L} \cong \Omega_8^+[m]$ for some $m \geq 8$.

Then there exists $(\mathcal{C}_1, \mathcal{C}_2) \in \mathfrak{P}[m]$ with $\mathcal{C}_i \leq \mathcal{L}$ and $z = z(\mathcal{C}_1)z(\mathcal{C}_2)$.

Proof. Recall from the proof of 1.3 that $\Omega_8^+[m] \cong HSpin_8^+[m]$, so (c) is a special case of (b). Thus $\mathcal{L}$ is tamely realized by a spin group or half spin group, as in 3.2. Then from 1.1 and the condition in (a) that $z = b$ when $n \equiv 0 \mod 4$ and $\epsilon = 1$, we have $z = z(\mathcal{C})z(\mathcal{C}')$ for $\mathcal{C} \in \mathfrak{L}(\mathcal{L})$ and $\mathcal{C}'$ the member of $\mathfrak{L}(\mathcal{L})$ paired to $\mathcal{C}$ as in 1.1. By 3.2 we may choose $z(\mathcal{C}) \in \mathcal{F}_z^f$ and $\mathcal{C} \in \mathrm{Comp}(C_{\mathcal{F}_z}(z(\mathcal{C}))$. Hence the lemma holds with $\mathcal{C}_1 = \mathcal{C}$ and $\mathcal{C}_2 = \mathcal{C}'$.

(3.11) *Assume t is an involution in $\mathcal{F}^f$ and $\mathcal{L} \in \mathrm{Comp}(\mathcal{F}_t)$ is in $Chev^*[m]$ for some $m \geq 16$. Let $\mathcal{C} \in \mathfrak{L}(\mathcal{L})$ such that $a = z(\mathcal{C}) \in \mathcal{F}_t^f$, and adopt the bar notation. Then one of the following holds:*

(1) $\bar{\mathcal{C}} \in Comp(\bar{\mathcal{F}})$.

(2) There exists $\mathcal{D}_1 \in \mathrm{Comp}(\bar{\mathcal{F}})$ isomorphic to $SL_2[m]$ such that $\mathcal{D}_2 = \mathcal{D}_1^{\bar{t}} \neq \mathcal{D}_1$ and $\bar{\mathcal{C}} = C_{\mathcal{D}_1\mathcal{D}_2}(\bar{t})$.

(3) There exists $\mathcal{D} \in \mathrm{Comp}(\bar{\mathcal{F}})$ such that $\bar{t}$ is nontrivial on $\mathcal{D}$, $\bar{\mathcal{C}} \in \mathrm{Comp}(C_{\mathcal{D}}(\bar{t}))$, and one of the following holds:

(i) $\mathcal{D}$ is isomorphic to $Spin_n^\epsilon[m]$ or $Spin_n^\epsilon[m/2]$ for some $n \geq 5$.

(ii) $\mathcal{D}$ is isomorphic to $HSpin_n^+[m]$ or $HSpin_n^+[m/2]$ for some $n \equiv 0 \mod 4$ with $n \geq 8$.

(iii) $\mathcal{D} \cong SL_2[2m]$ and $\bar{t}$ induces a field automorphism on $\mathcal{D}$.

(iv) $m = 16$, $\mathcal{D}/Z(\mathcal{D})$ is the 2-fusion system of $L_3(4)$, $\bar{a} \in \Phi(Z(\mathcal{D}))$, and $\bar{t}$ induces a field automorphism on $\mathcal{D}$.

(v) $m = 16$ and $\mathcal{D}$ is the 2-fusion system of $\hat{A}_n$ for some even n with $n \geq 8$.

Proof. As $\mathcal{C} \in \mathfrak{L}(\mathcal{L})$ with $a = z(\mathcal{C}) \in \mathcal{F}_t^f$, we have $SL_2[m] \cong \mathcal{C} \in \mathrm{Comp}(C_{\mathcal{F}_t}(a))$ from 3.2. Therefore the hypotheses of 3.8 are satisfied, so one of the three conclusions of that lemma is satisfied.

If 3.8.1 holds the conclusion (1) of our lemma is satisfied.

Suppose 3.8.3 holds. Then there exists $\mathcal{D}_1 \in \mathrm{Comp}(\bar{\mathcal{F}})$ such that $\mathcal{D}_2 = \mathcal{D}_1^{\bar{t}} \neq \mathcal{D}_1$ and $\bar{\mathcal{C}} = E(C_{\mathcal{D}_1\mathcal{D}_2}(\bar{t}))$ is a morphic image of $\mathcal{C}$. But $SL_2[m]$ has no proper covering, so $\mathcal{C} \cong \mathcal{D}_1$ and $\mathcal{D}_1\mathcal{D}_2 = \mathcal{D}_1 \times \mathcal{D}_2$ with $\bar{\mathcal{C}} = C_{\mathcal{D}_1\mathcal{D}_2}(\bar{t})$ a full diagonal subsystem. Thus conclusion (2) of our lemma is satisfied in this case.

Therefore we may assume 3.8.2 holds, and it remains to verify that one of the four subcases in conclusion (3) of our lemma holds. Observe $\bar{a} \in \bar{\mathcal{C}} \leq \mathcal{D}$, so $\bar{a} \in Z(\mathcal{D})$. Therefore $Z(\mathcal{D}) \neq 1$, so $\mathcal{D}$ is not Benson-Solomon. Hence by 3.4 and the discussion at the beginning of section 1, $\mathcal{D}$ is tamely realized by some known quasisimple group. Thus from 3.8.2 the hypothesis of 1.2 is satisfied with $\langle \bar{t} \rangle \mathcal{D}$ in the role of $\mathcal{F}$, so 1.2 completes the proof.

(3.12) *Assume some member of $\mathfrak{C}(\mathcal{F})$ is in $Chev^*[m]$ for some $m \geq 16$. Then either*

(0) $m = 16$ and there exists $SL_2[8] \cong \mathcal{D} \in \mathrm{Comp}_+(\mathcal{F}_j)$ where $j = z(\mathcal{D}) \in \mathcal{F}^f$, or there exists $\mathcal{D}$ intrinsic in $\mathfrak{C}(\mathcal{F})$ such that one of the following holds:

(1) $\mathcal{D} \cong SL_2[m]$ or $m > 16$ and $\mathcal{D} \cong SL_2[m/2]$.

(2) $\mathcal{D} \cong SL_2[2m]$.

(3) $\mathcal{D} \cong Spin_7[m]$.

(4) $\mathcal{D} \cong Spin_7[m/2]$.

(5) $m = 16$, $\mathcal{D}/Z(\mathcal{D})$ is the 2-fusion system of $L_3(4)$, and $\mathcal{I}(\mathcal{D}) \cap \Phi(Z(\mathcal{D})) \neq \varnothing$.

(6) $m = 16$ and $\mathcal{D}$ is the 2-fusion system of $\hat{A}_n$ for some even n with $n \geq 8$.

Proof. By hypothesis there is an involution $t \in \mathcal{F}^f$ and $\mathcal{L} \in Chev^*[m]$ for some $m \geq 16$ with $\mathcal{L} \in \mathrm{Comp}(\mathcal{F}_t)$. Therefore by 3.2 we may choose $\mathcal{C} \in \mathfrak{L}(\mathcal{L})$ with $a = z(\mathcal{C}) \in \mathcal{F}_t^f$ as in 3.11. Then by 3.11, one of the three conclusions of that lemma hold.

If 3.11.1 holds then conclusion (1) of our lemma holds. Suppose 3.11.2 holds. Then $(\mathcal{D}_1, \mathcal{D}_2) \in \mathfrak{P}[m]$, so by 3.9, conclusion (1) or (3) of our lemma holds.

Therefore we may assume that 3.11.3 holds. Suppose (3i) or (3ii) of 3.11 holds. Then by 3.10, there exists $(\mathcal{C}_1, \mathcal{C}_2) \in \mathfrak{P}[m_0]$ for $m_0 \in \{m, m/2\}$. Then by 3.9, conclusion (0), (1), (3) or (4) holds.

If 3.11.3.iii holds then conclusion (2) of our lemma holds, if 3.11.3.iv holds then conclusion (5) holds, and if 3.11.3.v holds then conclusion (6) holds. This completes the proof.

Remark 3.13. Observe that if some member of $\mathfrak{C}(\mathcal{F})$ is in $Chev^*[m]$ for some $m \geq 16$ then Theorem 3.3 holds by 3.12. Thus to complete the proof of Theorem 3.3 it remains to consider the case where, for each member of $\mathfrak{C}(\mathcal{F})$ in $Chev^*[m]$, we have $m = 8$. This case is treated in the next section.

Section 4. The basic theorem when $m = 8$

In this section we assume the following hypothesis:

Hypothesis 4.1. The hypothesis of Theorem 3.3 is satisfied and for each $\mathcal{L} \in \mathfrak{C}(\mathcal{F})$ with $\mathcal{L} \in Chev^*[m]$, we have $m = 8$. In addition neither conclusion (1) nor conclusion (2) of Theorem 3.3 is satisfied.

(4.2) *(1)* $\mathfrak{P}[8] = \varnothing$.

(2) There is no $\mathcal{L}$ intrinsic in $\mathfrak{C}(\mathcal{F})$ isomorphic to $HSpin_n^+[8]$ or to $Spin_n^\epsilon[8]$ with the distinguished involution b of 1.1 in $\mathcal{I}(\mathcal{L})$ in the latter case when $n \equiv 0 \mod 4$ and $\epsilon = 1$.

(3) There is no $\mathcal{L}$ intrinsic in $\mathfrak{C}(\mathcal{F})$ isomorphic to $Sp_4[8]$, $SL_4^\epsilon[8]$, or $\Omega_8^+[8]$.

Proof. As neither of the first two conclusions of Theorem 3.3 hold, part (1) follows from 3.9. Similarly (2) follows from (1) and 3.10, and then (2) and 3.10.c imply (3), keeping in mind that $Sp_4[8] = Spin_5[8]$ and $SL_4^\epsilon[8] = Spin_6^\epsilon[8]$.

(4.3) *There exists an involution $t \in \mathcal{F}^f$ and $\mathcal{L} \in \mathrm{Comp}(\mathcal{F}_t)$ such that one of the following holds:*

(1) $\mathcal{L}$ is an image of $Sp_n[8]$ for some $n \geq 4$, but not $PSp_4[8]$.

(2) $\mathcal{L}$ is an image of $SL_n^\epsilon[8]$ for some $n \geq 4$ and $\epsilon \in \{1, -1\}$, but not $L_4^\epsilon[8]$ or $\Omega_6^-[8]$.

(3) $\mathcal{L}$ is an image of $Spin_n^\epsilon[8]$ for some $n \geq 7$ and $\epsilon \in \{1, -1\}$, but not $P\Omega_k^\epsilon[8]$ for $k \in \{7, 8\}$.

(4) $\mathcal{L}$ is $F_4[8]$, $E_6^\epsilon[8]$, $E_7[8]$, $\tilde{E}_7[8]$, or $E_8[8]$.

Proof. By the hypothesis of Theorem 3.3 there exists an involution $t \in \mathcal{F}^f$ and $\mathcal{L} \in \mathrm{Comp}(\mathcal{F}_t)$ such that $\mathcal{L} \in Chev[large]$. Hence $\mathcal{L} \in Chev^*[m]$ for some m and by Hypothesis 4.1, $m = 8$. As $\mathcal{L}$ is large, $\mathcal{L}$ is not $L_3^\epsilon[8]$, $G_2[8]$, $P\Omega^\epsilon[8]$ with $5 \leq n \leq 8$, or $\Omega_6^-[8]$. Now the lemma follows using the discussion of coverings of the simple members of $Chev^*[8]$ in section 6.1 of [GLS3] or in the proof of 1.2.

Notation 4.4. Choose t and $\mathcal{L}$ as in 4.3. Then $\mathcal{L}$ is the 2-fusion system of some $L \in Chev^*(3)$. We proceed as in the proof of 1.2, so we set $L^* = L/Z(L)$. We focus on an involution $a \in \mathcal{F}_t^f$ and $\mathcal{C} \in \mathrm{Comp}(C_{\mathcal{L}}(a))$ such that $a \in T \in Syl(\mathcal{C})$. Indeed we choose $\mathcal{C} = \mathcal{F}_T(K)$ for some component K of $C_L(a)$. Notice that $\mathcal{C} \in \mathrm{Comp}(C_{\mathcal{F}_t}(a))$.

We essentially choose a as in 29.5 of [A7] and K as in the I_K column of Table 29.5 on page 456 of [A7]. In particular we will see that:

(4.5) *K is a spin group or half-spin group and $a = z(K_1)z(K_2)$ for suitable fundamental subgroups K_i of K and L.*

First suppose L^* is exceptional. Then L appears in 4.3.4 and we choose K to be $Spin_9(3)$, $Spin_{10}^{\epsilon}(3)$, $HSpin_{12}^{+}(3)$, $Spin_{12}^{+}(3)$, $HSpin_{16}^{+}(3)$ for L isomorphic to $F_4(3)$, $E_6^{\epsilon}(3)$, $E_7(3)$, $\tilde{E}_7(3)$, $E_8(3)$, respectively.

So assume L^* is classical. Then, as in the proof of 1.2, there is an n-dimensional vector space V over $F = \mathbf{F}_3$ (or $F = \mathbf{F}_9$ if L^* is unitary) such that L^* is the image of $\Omega = L^{\epsilon}(V)$, $Sp(V)$, or $\Omega^{\epsilon}(V)$. Let $\tilde{\Omega}$ be the universal covering group of L^*.

If L^* is $L_n^{\epsilon}(3)$ or $Sp_n(3)$, choose $a_0 \in \Omega$ so that $\dim([V, a_0]) = 4$ and a is the image of a_0 in L. Thus K is $SL_4^{\epsilon}(3) \cong Spin_6^{\epsilon}(3)$ or $Sp_4(3) \cong Spin_5(3)$ in the respective case. Therefore we may assume L^* is $P\Omega_n^{\epsilon}(3)$ with $n \geq 7$.

If $n = 7$ or 8 then by 4.3, L is $Spin_7(3)$, $Spin_8^{\epsilon}(3)$ or $\Omega_8^{+}(3)$, and in these cases we choose $K = L$ and a any involution in $Z(L)$ with $a = z(K_1)z(K_)$ for suitable fundamental subgroups K_i of K. Finally if $n \geq 9$ then we may choose a so that $a_0^* = a^*$ for $a_0 \in \Omega$ with $[V, a_0]$ of dimension 8 and sign 1; hence K is $\Omega_8^{+}(3)$ or $Spin_8^{+}(3)$.

In any event adopt the bar setup and notation of 3.6 with respect to the pair t, a.

(4.6) $\bar{\mathcal{C}} \notin \mathrm{Comp}(\bar{\mathcal{F}})$.

Proof. This follows from 4.2 and 4.5. Note if $\mathcal{C}$ is $Spin_n^{+}[8]$ with $n \equiv 0 \mod 4$ then $\bar{a}$ is the involution b of 1.1 as $a = z(K_1)z(K_2)$; thus we can indeed apply 4.2.

(4.7) *We may choose $(t, a, \mathcal{C})$ so that there is a component $\mathcal{D}$ of $\bar{\mathcal{F}}$ such that $\bar{t}$ is nontrivial on $\mathcal{D}$ and $\bar{\mathcal{C}} \in \mathrm{Comp}(C_{\mathcal{D}}(\bar{t}))$.*

Proof. Assume otherwise. Then by 3.8 and 4.6, conclusion (3) of 3.8 holds. Set $\mathcal{D}_1 = \mathcal{D}$ and $\mathcal{D}_2 = \mathcal{D}^{\bar{t}}$. Then $\mathcal{C}$ is a morphic image of $\mathcal{D}$, so from 4.5 either $\mathcal{C} \cong \mathcal{D}$ and $\mathcal{D}_1\mathcal{D}_2 = \mathcal{D}_1 \times \mathcal{D}_2$, or $\mathcal{C}$ is $HSpin_n^{+}[8]$ with $n \equiv 0 \mod 4$ and $\mathcal{D} \cong Spin_n^{+}[8]$. In any event we replace $(t, a, \mathcal{C})$ by $(t_1, a_1, \mathcal{D})$ where $t_1 = \bar{a}$ and a_1 is an involution in $Z(\mathcal{D}_1)$ satisfying 4.5. Observe that the second triple satisfies the constraints of the first, described in 4.5 and its proof. Adopt the bar setup for (t_1, a_1) and set $\bar{\mathcal{F}}_1 = C_{\mathcal{F}}(\bar{a}_1)$. By L-balance, $\bar{\mathcal{D}}_i \leq E(\bar{\mathcal{F}}_1)$ for $i = 1, 2$, so $\bar{t}_1 \in E(\bar{\mathcal{F}}_1)$ and hence $\bar{t}_1$ acts on each component of $\bar{\mathcal{F}}_1$. Therefore 3.8.3 does not hold in $\bar{\mathcal{F}}_1$, completing the proof of the lemma.

(4.8) *$\mathcal{D}$ is the 2-fusion system of some $J \in Chev^*(3)$ with $\bar{a} \in Z(J)$ and $\bar{a} \in \bar{K}$ a component of $C_J(\bar{t})$.*

Proof. By the hypothesis of Theorem 3.3, $\mathcal{D} \in \tilde{\mathcal{K}}$, so either $\mathcal{D}$ is the 2-fusion system of some quasisimple group J, or $\mathcal{D}$ is a Benson-Solomon system $\mathcal{F}_{Sol}(q)$. However in the latter case $Z(\mathcal{D}) = 1$ by Theorem 4.2 in [HL], contradicting $\bar{a} \in Z(\mathcal{D})$. Thus the former case holds. By 4.7 and 2.5.11 in [A5], $\bar{K}$ is a component of $C_J(\bar{t})$, so as $a \in K$, also $\bar{a} \in Z(J)$. Thus it remains to show that J can be chosen in $Chev^*(3)$. We argue as in the proof of 1.2, using the fact that $Z(J)$ is of even order to show $J \in Chev^*(p)$ for some odd prime p, and then as $m = 8$ we may choose $J \in Chev^*(3)$.

Set $J^* = J/Z(J)$. First by 4.5, $\bar{K}^*$ is not A_k for any k, so J^* is not alternating.

Suppose $J^* \in Chev(2)$. Then, as in the proof of 1.2, J^* appears in Table 6.1.3 in [GLS3] (or equivalently in 6.4.3 in [A5]) and $\bar{t}$ induces an outer automorphism on J^* with $\bar{K}^* \in Chev(2)$. As K^* is an orthogonal group over $\mathbf{F}_3$, it follows that K^* is $\Omega_5(3) \cong U_4(2)$. But inspecting the list in section 19 of [ASe] of centralizer of such automorphisms on groups in Table 6.1.3, we find none with a $U_4(2)$ component.

This leaves the case where J^* is sporadic. Then J^* is in Table 6.1.3 in [GLS3] (or equivalently in 6.4.2 in [A5]). Then inspecting the list of centralizers of automorphisms of such groups in [GLS3], we find no suitable components, completing the proof.

(4.9) *$J^* = J/Z(J)$ is classical.*

Proof. Assume otherwise. By 4.8, $J \in Chev^*(3)$, so if J^* is not classical then J is one of the exceptional groups in 4.3.4. Therefore as $|Z(J)|$ is even, $J \cong \tilde{E}_7(3)$. Inspecting the list of involution centralizers in $Aut(J^*)$ appearing in Table 4.5.1 in [GLS3] for a component $\bar{K}^*$ covered by a spin group, we conclude $K \cong Spin^+_{12}(3)$ and the fundamental subgroups of $\bar{K}$ are also fundamental subgroups of J. But in $J^* \cong E_7(2)$, we have $z(\bar{K}^*_1) \neq z(\bar{K}^*_2)$ for the fundamental subgroups K_i of K in 4.5, so that $\bar{a} = z(\bar{K}_1)z(\bar{K}_2) \notin Z(J)$, a contradiction.

Notation 4.10. By 4.9, J^* is a classical group, so we can adopt the corresponding notation in the proof of 1.2. That is V is an n-dimensional vector space over $F = \mathbf{F}_3$ (or $\mathbf{F}_9$ if J^* is unitary) such that J^* is an image of $\Omega = L^\epsilon(V)$, $Sp(V)$, or $\Omega^\epsilon(V)$. As usual $\tilde{\Omega}$ is the universal cover of Ω.

(4.11) *If J^* is $L^\epsilon_n(3)$ then $J \cong SL^\epsilon_8(3)$ and $K \cong \Omega^+_8(3)$.*

Proof. Assume otherwise. Inspection of the list in Table 4.5.1 of [GLS3] of centralizers of involutory automorphisms i^* of J^* such that $C_{J^*}(i^*)$ has a component $\bar{K}^*$ covered by a spin group, we conclude that either:

(a) $i \in \Omega$ with $\dim([V,i]) = 4$ and $K \cong SL_4^\epsilon(3)$ centralizes $C_V(i)$, or

(b) i induces a graph automorphism on Ω and $K \cong \Omega_n^\delta(3)$.

But in case (a) as $K \neq J$, we have $Z(K) \not\leq Z(J)$, a contradiction. On the other hand in case (b), it follows from 4.5 that $n = 8$ and $\delta = 1$.

(4.12) *If J^* is $PSp_n(3)$ then $J \cong Sp_8(3)$ and $K \cong SL_4^\epsilon(3)$.*

Proof. We proceed as in the proof of the previous lemma and conclude that either

(a) $i \in \Omega$ with $\dim([V,i]) = 4$ and $K \cong Sp_4(3)$ centralizes $C_V(i)$, or

(b) $n = 8$, i induces an outer automorphism on Ω and $K \cong SL_4^\epsilon(3)$.

In case (a) as $\bar{a} \in Z(\bar{K}) \cap Z(J)$, we conclude that $K = J$, a contradiction.

(4.13) *J^* is not $P\Omega_n^\epsilon(3)$.*

Proof. Assume otherwise. If n is odd then as $Z(J) \neq 1$, we have $J \cong Spin_n(3)$, contrary to 4.2.2. Thus n is even, and by a similar argument, $J = \Omega \cong \Omega_n^\epsilon(3)$, with $\epsilon = -1$ if $n \equiv 2 \mod 4$ and $\epsilon = 1$ if $n \equiv 0 \mod 4$. Then by 4.2.3, $n \neq 8$.

As usual we inspect the list of centralizers of involutory automorphisms i^* of J^* and conclude (since $n \neq 8$) that i is of type $i(r)$ or $i(s,\alpha)$, for r odd or s even, and $K \cong \Omega_r(3)$ or $\Omega_s^\epsilon(3)$ centralizes $C_V(i)$. But now as $K \neq J$, $Z(\bar{K}) \not\leq Z(J)$, a contradiction.

(4.14) *(1) J is not $Sp_8(3)$.*

(2) J is $SL_8^\epsilon(3)$ and $K \cong \Omega_8^+(3)$.

Proof. Suppose J is $Sp_8(3)$. Replace (t,a) by (t_1,a_1) where $t_1 = \bar{a}$ and $a_1 \in J$ with $\dim([V,a_1]) = 4$. Thus $K_1 \cong Sp_4(3)$. But now 4.10-4.13 applied to (t_1,a_1) yield a contradiction. This establishes (1). Then (1) and 4.10-4.13 imply (2).

(4.15) *J is not $SL_8^\epsilon(3)$.*

Proof. Suppose J is $SL_8^\epsilon(3)$. Replace (t,a) by (t_1,a_1) where $t_1 = \bar{a}$ and $a_1 \in J$ with $\dim([V,a_1]) = 4$. Thus $K_1 \cong SL_4^\epsilon(3)$. But now by 4.11-4.13, J_1 is $Sp_8(3)$, contrary to 4.14.1.

Note that 4.14.2 and 4.15 supply a contradiction. This contradiction shows that Hypothesis 4.1 is never satisfied. Together with Remark 3.13, this completes the proof of Theorem 3.3. Observe that Theorem 3.3 is just Walter's Basic Theorem.

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Pasadena, California 91125